150C Causal Inference

Instrumental Variables: Traditional Perspective

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Outline

Effect of Training in JTPA

R Code

```
> d <- read.dta("jtpa.dta")
> summary(lm(earnings~training,data=d))
Call:
lm(formula = earnings ~ training, data = d)

Residuals:
   Min      1Q Median      3Q     Max
-17396 -13587  -4955   8776 141155

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  14605.1      209.8  69.624   <2e-16 ***
training      2791.1      318.6   8.761   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 16710 on 11202 degrees of freedom
Multiple R-squared: 0.006806,        Adjusted R-squared: 0.006717
F-statistic: 76.76 on 1 and 11202 DF,  p-value: < 2.2e-16
```

Motivation for Instrumental Variables: Non-Compliance

Problem

- *Often we cannot force subjects to take specific treatments*
- *Units choosing to take the treatment may differ in unobserved characteristics from units that refrain from doing so*

Example: Non-compliance in JTPA Experiment

	Not Enrolled in Training	Enrolled in Training	Total
Assigned to Control	3,663	54	3,717
Assigned to Training	2,683	4,804	7,487
Total	6,346	4,858	11,204

Two Views on Instrumental Variables

1. Traditional Econometric Framework
    - Constant treatment effects
    - Linearity in case of a multivalued treatment
2. Potential Outcome Model of IV
    - Heterogeneous treatment effects
    - Focus in Local Average Treatment Effect (LATE)

Outline

1. Motivation

2. Traditional Instrumental Variable Framework
   - IV Assumptions
   - First Stage Effect
   - Reduced Form/Intent-to-treat Effect
   - IV Effect: Wald Estimator and 2SLS
   - IV Effect: Multivariate Case

3. Problems with IV
   - Weak Instruments
   - Failure of Exogeneity

Recall the Omitted variable bias

- True model: $Y = \alpha_0 + \alpha_1 D + u_2$
  - D is the treatment variable (e.g. training)
  - D may be endogenous so that $Cov[D, u_2] \neq 0$

- Recall that the OLS estimator for α_1 is given by:

$$\hat{\alpha}_{1,OLS} = \frac{Cov[Y, D]}{V[D]} =$$

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$$\hat{\alpha}_{1,OLS} = \frac{\alpha_1 Cov[D, D] + Cov[D, u_2]}{Cov[D, D]}$$

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$$\hat{\alpha}_{1,OLS} = \frac{Cov[Y, D]}{V[D]} = \frac{Cov[\alpha_0 + \alpha_1 D + u_2, D]}{Cov[D, D]}$$

$$\hat{\alpha}_{1,OLS} = \frac{\alpha_1 Cov[D, D] + Cov[D, u_2]}{Cov[D, D]}$$

$$\hat{\alpha}_{1,OLS} = \alpha_1 + \frac{Cov[D, u_2]}{Cov[D, D]}$$

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$$\hat{\alpha}_{1,OLS} = \frac{\alpha_1 Cov[D, D] + Cov[D, u_2]}{Cov[D, D]}$$

$$\hat{\alpha}_{1,OLS} = \alpha_1 + \frac{Cov[D, u_2]}{Cov[D, D]}$$

$$E[\hat{\alpha}_{1,OLS}] = \alpha_1 + E[\frac{Cov[D, u_2]}{Cov[D, D]}]$$

so bias depends on correlation between u and D

Outline

Instrumental Variable Estimator Assumptions

Imagine we have two equations:

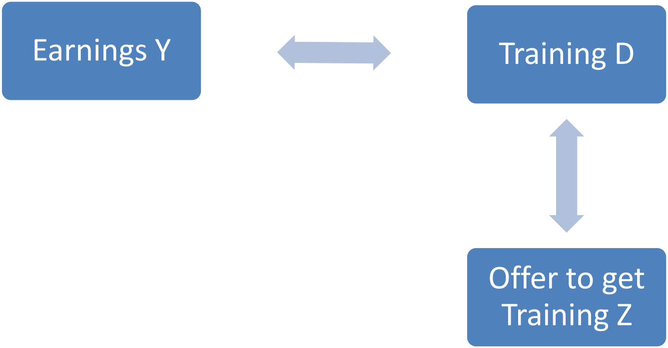
- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
  - Z is our instrumental variable (e.g. randomized encouragement)
  - π_1 is effect of Z on D

A valid instrument needs to satisfy three assumptions:

1. $\pi_1 \neq 0$ so Z affects the endogenous treatment D (called first stage or relevance)
2. Z is as good as randomly assigned so $Cov[u_1, Z] = 0$
3. Z satisfies the exclusion restriction, i.e. Z has no effect on Y other than through D. In other words, Z has no independent effect on Y and that is why it does not appear in the second stage equation and we assume $Cov[u_2, Z] = 0$

Which of these is testable?

Instrumental Variable Estimator Assumptions



Instrumental Variable Estimator Assumptions

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

Based on these IV assumptions we can identify three effects:

1. The first stage effect: Effect of Z on D.
2. Reduced from or intent-to-treat effect (ITT): Effect of Z on Y.
3. The instrumental variable treatment effect: Effect of D on Y, using only the exogenous variation in D that is induced by Z.

Outline

First Stage Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

First stage effect: Z on D

$$\hat{\pi}_1 = \frac{Cov[D, Z]}{V[Z]}$$

First Stage Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

First stage effect: Z on D

$$\hat{\pi}_1 = \frac{Cov[D, Z]}{V[Z]} = \frac{Cov[\pi_0 + \pi_1 Z + u_1, Z]}{Cov[Z, Z]}$$

First Stage Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

First stage effect: Z on D

$$\hat{\pi}_1 = \frac{Cov[D, Z]}{V[Z]} = \frac{Cov[\pi_0 + \pi_1 Z + u_1, Z]}{Cov[Z, Z]}$$

$$\hat{\pi}_1 = \frac{\pi_1 Cov[Z, Z] + Cov[Z, u_1]}{Cov[Z, Z]}$$

First Stage Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

First stage effect: Z on D

$$
\begin{aligned}
\hat{\pi}_1 &= \frac{Cov[D, Z]}{V[Z]} = \frac{Cov[\pi_0 + \pi_1 Z + u_1, Z]}{Cov[Z, Z]} \\
\hat{\pi}_1 &= \frac{\pi_1 Cov[Z, Z] + Cov[Z, u_1]}{Cov[Z, Z]} \\
\hat{\pi}_1 &= \pi_1 + \frac{Cov[Z, u_1]}{Cov[Z, Z]}
\end{aligned}
$$

First Stage Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

First stage effect: Z on D

$$\hat{\pi}_1 = \frac{Cov[D,Z]}{V[Z]} = \frac{Cov[\pi_0 + \pi_1 Z + u_1, Z]}{Cov[Z,Z]}$$

$$\hat{\pi}_1 = \frac{\pi_1 Cov[Z,Z] + Cov[Z,u_1]}{Cov[Z,Z]}$$

$$\hat{\pi}_1 = \pi_1 + \frac{Cov[Z,u_1]}{Cov[Z,Z]}$$

$$E[\hat{\pi}_1] = \pi_1 + E[\frac{Cov[Z,u_1]}{Cov[Z,Z]}]$$

First Stage Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

First stage effect: Z on D

$$\hat{\pi}_1 = \frac{Cov[D, Z]}{V[Z]} = \frac{Cov[\pi_0 + \pi_1 Z + u_1, Z]}{Cov[Z, Z]}$$

$$\hat{\pi}_1 = \frac{\pi_1 Cov[Z, Z] + Cov[Z, u_1]}{Cov[Z, Z]}$$

$$\hat{\pi}_1 = \pi_1 + \frac{Cov[Z, u_1]}{Cov[Z, Z]}$$

$$E[\hat{\pi}_1] = \pi_1 + E[\frac{Cov[Z, u_1]}{Cov[Z, Z]}] = \pi_1$$

$\hat{\pi}_1$ is consistent since $Cov[u_1, Z] = 0$

First Stage Effect in JTPA

First stage effect: Z on D: $\hat{\pi}_1 = \frac{Cov[D,Z]}{V[Z]}$

R Code

```
> cov(d[,c("earnings","training","assignmt")])
             earnings    training    assignmt
earnings 2.811338e+08 685.5254685 257.0625061
training 6.855255e+02   0.2456123   0.1390407
assignmt 2.570625e+02   0.1390407   0.221713
```

R Code

```
> 0.1390407/0.2217139
[1] 0.6271177
```

First Stage Effect in JTPA

R Code

```
> summary(lm(training~assignmt,data=d))

Call:
lm(formula = training ~ assignmt, data = d)

Residuals:
     Min       1Q   Median       3Q      Max
-0.64165 -0.01453 -0.01453  0.35835  0.98547

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.014528   0.006529   2.225   0.0261 *
assignmt    0.627118   0.007987  78.522   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.398 on 11202 degrees of freedom
Multiple R-squared: 0.355,        Adjusted R-squared: 0.355
F-statistic:  6166 on 1 and 11202 DF,  p-value: < 2.2e-1
```

Outline

Reduced Form/Intent-to-treat Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

Reduced Form/Intent-to-treat Effect: Z on Y: Plug first into second stage:

$$Y = \alpha_0 + \alpha_1(\pi_0 + \pi_1 Z + u_1) + u_2$$

Reduced Form/Intent-to-treat Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

Reduced Form/Intent-to-treat Effect: Z on Y: Plug first into second stage:

$$\begin{aligned} Y &= \alpha_0 + \alpha_1(\pi_0 + \pi_1 Z + u_1) + u_2 \\ Y &= (\alpha_0 + \alpha_1 \pi_0) + (\alpha_1 \pi_1) Z + (\alpha_1 u_1 + u_2) \end{aligned}$$

Reduced Form/Intent-to-treat Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

Reduced Form/Intent-to-treat Effect: Z on Y: Plug first into second stage:

$$\begin{aligned} Y &= \alpha_0 + \alpha_1(\pi_0 + \pi_1 Z + u_1) + u_2 \\ Y &= (\alpha_0 + \alpha_1 \pi_0) + (\alpha_1 \pi_1)Z + (\alpha_1 u_1 + u_2) \\ Y &= \gamma_0 + \gamma_1 Z + u_3 \end{aligned}$$

where $\gamma_0 = \alpha_0 + \alpha_1 \pi_0$, $\gamma_1 = \alpha_1 \pi_1$, and $u_3 = \alpha_1 u_1 + u_2$.

Reduced Form/Intent-to-treat Effect

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where $\gamma_0 = \alpha_0 + \alpha_1 \pi_0$, $\gamma_1 = \alpha_1 \pi_1$, and $u_3 = \alpha_1 u_1 + u_2$. Note that

$$\hat{\gamma}_1 = \frac{Cov[Y, Z]}{Cov[Z, Z]} = \frac{Cov[\gamma_0 + \gamma_1 Z + u_3, Z]}{Cov[Z, Z]}$$

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where $\gamma_0 = \alpha_0 + \alpha_1\pi_0$, $\gamma_1 = \alpha_1\pi_1$, and $u_3 = \alpha_1 u_1 + u_2$. Note that

$$\begin{aligned} \hat{\gamma}_1 &= \frac{Cov[Y, Z]}{Cov[Z, Z]} = \frac{Cov[\gamma_0 + \gamma_1 Z + u_3, Z]}{Cov[Z, Z]} \\ E[\hat{\gamma}_1] &= \gamma_1 + E[\frac{Cov[Z, u_3]}{Cov[Z, Z]}] = \gamma_1 \end{aligned}$$

$\hat{\gamma}_1$ is consistent since $Cov[u_1, Z] = 0$ and $Cov[u_2, Z] = 0$ implies $Cov[u_3, Z] = 0$

Reduced Form/Intent-to-treat Effect

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> summary(lm(earnings~assignmt,data=d))

Call:
lm(formula = earnings ~ assignmt, data = d)

Residuals:
   Min     1Q Median     3Q    Max
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Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  15040.5      274.9  54.716  < 2e-16 ***
assignmt      1159.4      336.3   3.448 0.000567 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1

Residual standard error: 16760 on 11202 degrees of freedom
Multiple R-squared: 0.00106,	Adjusted R-squared: 0.000971
F-statistic: 11.89 on 1 and 11202 DF,  p-value: 0.000566
```

Outline

Instrumental Variable Effect: Wald Estimator

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

IV Effect: X on Y using exogenous variation in D that is induced by Z. Recall

$$Y = (\alpha_0 + \alpha_1 \pi_0) + (\alpha_1 \pi_1) Z + (\alpha_1 u_1 + u_2)$$

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IV Effect: X on Y using exogenous variation in D that is induced by Z. Recall

$$
\begin{aligned}
Y &= (\alpha_0 + \alpha_1 \pi_0) + (\alpha_1 \pi_1) Z + (\alpha_1 u_1 + u_2) \\
Y &= \gamma_0 + \gamma_1 Z + u_3
\end{aligned}
$$

where $\gamma_0 = \alpha_0 + \alpha_1 \pi_0$, $\gamma_1 = \alpha_1 \pi_1$, and $u_3 = \alpha_1 u_1 + u_2$. Given this, we can identify α_1:

$$
\alpha_1 = \frac{\gamma_1}{\pi_1}
$$

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where $\gamma_0 = \alpha_0 + \alpha_1 \pi_0$, $\gamma_1 = \alpha_1 \pi_1$, and $u_3 = \alpha_1 u_1 + u_2$. Given this, we can identify α_1:

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\alpha_1 = \frac{\gamma_1}{\pi_1} =
$$

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$$
\alpha_1 = \frac{\gamma_1}{\pi_1} = \frac{\text{Effect of Z on Y}}{\text{Effect of Z on D}} = \frac{Cov[Y, Z]/Cov[Z, Z]}{Cov[D, Z]/Cov[Z, Z]} = \frac{Cov[Y, Z]}{Cov[D, Z]}
$$

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Instrumental Variable Effect: Wald Estimator

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$$\begin{aligned} \alpha_1 &= \frac{\gamma_1}{\pi_1} = \frac{\text{Effect of Z on Y}}{\text{Effect of Z on D}} = \frac{Cov[Y, Z]/Cov[Z, Z]}{Cov[D, Z]/Cov[Z, Z]} = \frac{Cov[Y, Z]}{Cov[D, Z]} \\ \hat{\alpha}_1 &= \frac{Cov[\alpha_0 + \alpha_1 D + u_2, Z]}{Cov[D, Z]} = \frac{\alpha_1 Cov[D, Z] + Cov[u_2, Z]}{Cov[D, Z]} = \end{aligned}$$

Instrumental Variable Effect: Wald Estimator

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

IV Effect: X on Y using exogenous variation in D that is induced by Z. Recall

$$\begin{aligned} Y &= (\alpha_0 + \alpha_1 \pi_0) + (\alpha_1 \pi_1) Z + (\alpha_1 u_1 + u_2) \\ Y &= \gamma_0 + \gamma_1 Z + u_3 \end{aligned}$$

where $\gamma_0 = \alpha_0 + \alpha_1 \pi_0$, $\gamma_1 = \alpha_1 \pi_1$, and $u_3 = \alpha_1 u_1 + u_2$. Given this, we can identify α_1:

$$\begin{aligned} \alpha_1 &= \frac{\gamma_1}{\pi_1} = \frac{\text{Effect of Z on Y}}{\text{Effect of Z on D}} = \frac{Cov[Y,Z]/Cov[Z,Z]}{Cov[D,Z]/Cov[Z,Z]} = \frac{Cov[Y,Z]}{Cov[D,Z]} \\ \hat{\alpha}_1 &= \frac{Cov[\alpha_0 + \alpha_1 D + u_2, Z]}{Cov[D,Z]} = \frac{\alpha_1 Cov[D,Z] + Cov[u_2,Z]}{Cov[D,Z]} = \alpha_1 + \frac{Cov[u_2,Z]}{Cov[D,Z]} \\ E[\hat{\alpha}_1] &= \alpha_1 + E[\frac{Cov[u_2,Z]}{Cov[D,Z]}] \end{aligned}$$

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IV Effect: X on Y using exogenous variation in D that is induced by Z. Recall

$$\begin{aligned} Y &= (\alpha_0 + \alpha_1 \pi_0) + (\alpha_1 \pi_1) Z + (\alpha_1 u_1 + u_2) \\ Y &= \gamma_0 + \gamma_1 Z + u_3 \end{aligned}$$

where $\gamma_0 = \alpha_0 + \alpha_1 \pi_0$, $\gamma_1 = \alpha_1 \pi_1$, and $u_3 = \alpha_1 u_1 + u_2$. Given this, we can identify α_1:

$$\begin{aligned} \alpha_1 &= \frac{\gamma_1}{\pi_1} = \frac{\text{Effect of Z on Y}}{\text{Effect of Z on D}} = \frac{Cov[Y,Z]/Cov[Z,Z]}{Cov[D,Z]/Cov[Z,Z]} = \frac{Cov[Y,Z]}{Cov[D,Z]} \\ \hat{\alpha}_1 &= \frac{Cov[\alpha_0 + \alpha_1 D + u_2, Z]}{Cov[D,Z]} = \frac{\alpha_1 Cov[D,Z] + Cov[u_2,Z]}{Cov[D,Z]} = \alpha_1 + \frac{Cov[u_2,Z]}{Cov[D,Z]} \\ E[\hat{\alpha}_1] &= \alpha_1 + E[\frac{Cov[u_2,Z]}{Cov[D,Z]}] = \alpha_1 \end{aligned}$$

$\hat{\alpha}_1$ is consistent if $Cov[u_2, Z] = 0$. What if $\pi_1 = 0$?

Instrumental Variable Effect: Wald Estimator

Instrumental Variable Effect: $\alpha_1 = \dfrac{\text{Effect of Z on Y}}{\text{Effect of Z on D}} = \dfrac{Cov[Y,Z]}{Cov[D,Z]}$

R Code

```
> cov(d[,c("earnings","training","assignmt")])
              earnings    training    assignmt
earnings 2.811338e+08 685.5254685 257.0625061
training 6.855255e+02   0.2456123   0.1390407
assignmt 2.570625e+02   0.1390407   0.221713
```

R Code

```
> 257.0625061/0.1390407
[1] 1848.829
```

Instrumental Variable Effect: Two Stage Least Squares

The instrumental variable estimator:

$$\alpha_1 = \frac{\gamma_1}{\pi_1} = \frac{Cov[Y, Z]}{Cov[D, Z]}$$

is numerically equivalent to the following two step procedure:

1. Fit first stage and obtain fitted values $\hat{D} = \hat{\pi}_0 + \hat{\pi}_1 Z$
2. Plug into second stage:

$$\begin{aligned} Y &= \alpha_0 + \alpha_1 \hat{D} + u_2 \\ Y &= \alpha_0 + \alpha_1 (\hat{\pi}_0 + \hat{\pi}_1 Z) + u_2 \\ Y &= (\alpha_0 + \alpha_1 \hat{\pi}_0) + \alpha_1 (\hat{\pi}_1 Z) + u_2 \end{aligned}$$

- α_1 is solely identified based on variation in D that comes from Z
- Point estimates from second regression are equivalent to IV estimator, the standard errors are not quite correct since they ignore the estimation uncertainty in $\hat{\pi}_0$ and $\hat{\pi}_1$.

Instrumental Variable Effect: Two Stage Least Squares

R Code

```
> training_hat <- lm(training~assignmt,data=d)$fitted
> summary(lm(earnings~training_hat,data=d))

Call:
lm(formula = earnings ~ training_hat, data = d)

Residuals:
   Min     1Q Median     3Q    Max
-16200 -13803  -4817   8950 139560

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   15013.6      281.3  53.375  < 2e-16 ***
training_hat   1848.8      536.2   3.448 0.000567 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 16760 on 11202 degrees of freedom
Multiple R-squared: 0.00106,       Adjusted R-squared: 0.000971
F-statistic: 11.89 on 1 and 11202 DF,  p-value: 0.0005669
```

Instrumental Variable Effect: Two Stage Least Squares

R Code

```
> library(AER)
> summary(ivreg(earnings ~ training | assignmt,data = d))
Call:
ivreg(formula = earnings ~ training | assignmt, data = d)
Residuals:
   Min     1Q Median     3Q    Max
-16862 -13716  -4943   8834 140746
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  15013.6      280.6  53.508  < 2e-16 ***
training      1848.8      534.9   3.457 0.000549 ***
---
Residual standard error: 16720 on 11202 degrees of freedom
Multiple R-Squared: 0.00603,       Adjusted R-squared: 0.005941
Wald test: 11.95 on 1 and 11202 DF,  p-value: 0.0005491
```

Outline

IV Estimator: Multivariate Case

- Let $\mathbf{X} = [1, X_1, ..., X_K, D]$ and $\mathbf{Z} = [1, X_1, ..., X_K, Z]$.
- Second Stage: $Y = \mathbf{X}\alpha + u_2$ with $\alpha = [\alpha_0, \alpha_1, ..., \alpha_K, \alpha_D]$
- First Stage: $D = \mathbf{Z}\pi + u_1$ with with $\pi = [\pi_0, \pi_1, ..., \pi_K, \pi_Z]$
- Identification: $Cov[\mathbf{Z}, u_1] = 0$, $Cov[\mathbf{Z}, u_2] = 0$, and $\pi_Z \neq 0$ (non-zero partial effect of Z on D)

The multivariate IV estimator is consistent:

$$
\begin{aligned}
\hat{\alpha}_{IV} &= (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'Y \\
\hat{\alpha}_{IV} &= (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'(\mathbf{X}\alpha + u_2) \\
\hat{\alpha}_{IV} &= (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'\mathbf{X}\alpha + (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'u_2 \\
\hat{\alpha}_{IV} &= \alpha + (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'u_2 \\
E[\hat{\alpha}_{IV}] &= \alpha + E[(\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'u_2] = \alpha
\end{aligned}
$$

2SLS Estimator: Multivariate Case

1. First stage regression to get fitted values

$$\begin{aligned} D &= \mathbf{Z}\pi + u_1 \Rightarrow \hat{\pi} = (\mathbf{Z}\mathbf{Z}')^{-1}\mathbf{Z}'D \\ \hat{D} &= \mathbf{Z}\hat{\pi} = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'D = \mathbf{P}_z D \end{aligned}$$

2SLS Estimator: Multivariate Case

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$$
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\hat{D} &= \mathbf{Z}\hat{\pi} = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'D = \mathbf{P}_z D
\end{aligned}
$$

2. Regress fitted values on Y

$$
Y = \hat{D}\alpha_{2SLS} + u_3
$$

We can show that:

$$
\begin{aligned}
\alpha_{2SLS} &= (\hat{D}'\hat{D})^{-1}\hat{D}'Y \\
&= (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'Y = \alpha_{IV}
\end{aligned}
$$

Instrumental Variable Effect: Two Stage Least Squares

R Code

```
> summary(ivreg(earnings ~ training + prevearn + sex + age + married
+              | prevearn + sex + age + married +assignmt,data = d))
Call:
ivreg(formula = earnings ~ training + prevearn + sex + age +
    married | prevearn + sex + age + married + assignmt, data = d)
Residuals:
   Min      1Q Median     3Q    Max
-58052 -10916  -4050   8316 117239

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.162e+04  6.042e+02  19.238  < 2e-16 ***
training     1.927e+03  4.998e+02   3.855 0.000116 ***
prevearn     1.270e+00  3.885e-02  32.675  < 2e-16 ***
sex          3.760e+03  3.053e+02  12.316  < 2e-16 ***
age         -9.592e+01  1.543e+01  -6.215  5.3e-10 ***
married      2.707e+03  3.488e+02   7.760  9.2e-15 ***
---
Residual standard error: 15600 on 11198 degrees of freedom
Multiple R-Squared: 0.1348,        Adjusted R-squared: 0.1344
Wald test:   335 on 5 and 11198 DF,  p-value: < 2.2e-16
```

Multiple Instruments

- 2SLS estimator can be used to combine multiple instruments for the same endogeneous variable. Strong assumptions needed:
  - Each instrument captures the same effect
  - Exogeneity holds for all instruments

$$D = X\beta + Z_1\pi_1 + Z_2\pi_2 + ... + Z_k\pi_k + u_1$$

where $Cov(Z_j, u_1) = 0$ and $Cov(Z_j, u_2) = 0$ for all $j = 1, .., k$.

- Need at least as many instruments as endogenous regressors:
  - Let k be number of endogenous regressors and m number of instruments
  - Exactly or just identified case: $m = k$
  - Overidentified case: $m > k$
  - Underidentified case: $m < k$

Outline

Judging the Credibility of IV Estimates

- The probability limit of the IV estimator is given by:

$$plim\, \hat{\alpha}_{D,IV} = \alpha_D + \frac{Corr(Z, u_2)}{Corr(Z, D)} \frac{\sigma^{u_2}}{\sigma^D}$$

  so to obtain consistent estimates the instrument Z must be:

- Relevant: $Cov(Z, D) \neq 0$ (testable)
  - If $Cov(Z, D)$ is small, the instrument is weak. We get consistency in asymptotia, but in small (finite) samples we can get strong bias even if instrument is perfectly exogenous
- Exogenous: $Cov(Z, u_2) = 0$ (untestable)
  - If Z has an independent effect on Y other than through D we have $Cov(Z, u_2) \neq 0$ and estimates are inconsistent
  - Even small violations can lead to significant large sample bias unless instruments are strong
- Failure of either condition is a problem! But both conditions can be hard to satisfy at the same time. There often is a tradeoff.

Instrumental Variable Examples

Study	Outcome	Treatment	Instrument
Angrist and Evans (1998)	Earnings	More than 2 Children	Multiple Second Birth (Twins)
Angrist and Evans (1998)	Earnings	More than 2 Children	First Two Children are Same Sex
Levitt (1997)	Crime Rates	Number of Policemen	Mayoral Elections
Angrist and Krueger (1991)	Earnings	Years of Schooling	Quarter of Birth
Angrist (1990)	Earnings	Veteran Status	Vietnam Draft Lottery
Miguel, Satyanath and Sergenti (2004)	Civil War Onset	GDP per capita	Lagged Rainfall
Acemoglu, Johnson and Robinson (2001)	Economic performance	Current Institutions	Settler Mortality in Colonial Times
Cleary and Barro (2006)	Religiosity	GDP per capita	Distance from Equator

Outline

Exogenous, but weak Instruments

- In contrast to OLS, the IV estimator is not unbiased in small (finite) samples even when instrument is perfectly exogenous

Exogenous, but weak Instruments

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Exogenous, but weak Instruments

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- Finite sample bias can be considerable (e.g., 20 - 30%), even when the sample size is over 100,000 if the instrument is weak
- Relative bias of $\alpha_{D,IV}$ versus $\alpha_{D,OLS}$ is approximately $1/F$ where F is the F-statistic for testing H_0: $\pi_Z = 0$, i.e. partial effect of Z on D is zero (or against joint zero for multiple instruments)

Testing For Relevance

R Code

```
> library(lmtest)
> fs1 <- lm(training~prevearn + sex + age + married +assignmt,data=d)
> fs2 <- lm(training~prevearn + sex + age + married,data=d)
> waldtest(fs1, fs2)
Wald test

Model 1: training ~ prevearn + sex + age + married + assignmt
Model 2: training ~ prevearn + sex + age + married
  Res.Df Df      F    Pr(>F)
1  11198
2  11199 -1 6158.8 < 2.2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1
```

Exogenous, but weak Instruments

- Adding instruments increases the relevance of the instrument set (increases the first stage F)
- But too many instruments increases small sample bias (compared to few instruments) and also call into doubt the exclusion restrictions
- Best to have single, strong instrument
- There are more complex competitors to 2SLS:
  - Limited Information Maximum Likelihood (LIML) estimation
  - Jackknife instrumental variables
  - Imbens and Rosenbaum (2005) robust IV.
- Small sample studies suggest that LIML and robust IV may be superior to 2SLS in small samples (but remains open area of research)

Outline

Failure of Exogeneity

- Recall the probability limit:

$$plim\,\hat{\alpha}_{D,IV} = \alpha_D + \frac{Corr(Z, u_2)}{Corr(Z, D)} \frac{\sigma_{u_2}}{\sigma_D}$$

- In general we get inconsistent estimates if $Corr(Z, u_2) \neq 0$. This large sample bias can often be considerable but is hard to quantify precisely because it depends on unobservables
- If the instrument is stronger, large sample bias can be attenuated, but often magnitude of $Corr(Z, u_2)$ dominates
- The best we can often do is often to be skeptical and to make sure exogeneity is highly plausible in the setting to which we apply IV
- Sensitivity analysis:
    - Is the instrument plausibly exogenous or can it be easily predicted from covariates?
    - Formal sensitivity tests
        - E.g. Stata code from "Plausibly Exogenous" (Hanson et. al, 2009)
        - R code from Wand (2002)

Failure of Exogeneity

- Does a randomly assigned instrument Z always satisfy $Cov(Z, u_2) = 0$?

Failure of Exogeneity

- Does a randomly assigned instrument Z always satisfy $Cov(Z, u_2) = 0$?
- No! Encouragement may still have independent effect on outcome other than through the treatment
- When designing an encouragement experiment we need to be careful to design encouragements so that they are relevant, but also narrowly targeted to only create variation in treatment intake
- SUTVA may be a concern as well

Conclusion

- IV works only under very specific circumstances (e.g. well designed encouragement design experiments)
- Often, it will be difficult to find instruments that are both relevant (strong enough) and exogenous
- Violations of assumptions can lead to large biases and estimation theory is complicated
- So far, we have assumed constant treatment effects α_D which seems unrealistic in most settings. Often an instrument affects only a subpopulation of interest and we have little information about treatment effects for other units that may not be affected by the instrument at all.
- Next we'll discuss modern IV with heterogeneous potential outcomes